Naam: Thomas ten Cate
Adres: Borgweg 82
Postcode en
Woonplaats: 9616 TK Scharmer
Studentnummer: 1350802
Studierichting: Informatica & Wiskunde
Jaar van eerste inschrijving: 2002
Bladnr.: 1
Tentamen: Fourier theorie
Datum: 05-11-2003
Naam docent: de Snoo

1a $f_n : [0,1] \to \mathbb{C}$ convergeert uniform, dus:

$\lim_{n\to\infty} \|f_n\|_\infty = 0$

oftewel:

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ z.d.d. $n \geq N \Rightarrow |f_n(x)| \leq \varepsilon \ \forall x \in [0,1]$

Hiervan is het volgende een gevolg:

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ z.d.d. $n \geq N \Rightarrow |f_n(\frac{1}{n})| \leq \varepsilon$

Immers, als het geldt voor alle $x \in [0,1]$ geldt het ook voor $x = \frac{1}{n}$ als $n \geq 1$

Dus hier staat:

$\lim_{n\to\infty} f_n(\frac{1}{n}) = 0$

b $f_n(x) = nx(1-x)^n$

$f_n(\frac{1}{n}) = n\frac{1}{n}(1-\frac{1}{n})^n = (1-\frac{1}{n})^n$

$\lim_{n\to\infty} f_n(\frac{1}{n}) = \frac{1}{e}$

Deze limiet is niet 0, dus kan $f_n(x)$ volgens (a) nooit uniform convergent zijn. $(p \to q) \Leftrightarrow (\neg q \to \neg p)$

c $f_n(x) = nx(1-x)^n$

$f_n : [a,1] \to \mathbb{R}$ met $0 < a < 1$

Als $a \leq x \leq 1$ geldt $1-1 \leq 1-x \leq 1-a$ dus $0 \leq 1-x \leq 1-a$

Omdat $0 < a$ geldt $1-a < 1$. Definieer nu $b = 1-x$:

$\lim_{n\to\infty} \|f_n\|_\infty = \lim_{n\to\infty} \sup_{x\in[a,1]} nx(1-x)^n = \lim_{n\to\infty} \sup_{b\in[0,1-a]} n(1-b)b^n \leq$

~~take b = a~~

$\lim \sup nb^n \leq \lim \sup nb^n = 0$

Dus $f_n(x)$ convergeert uniform naar 0.

2a $f \in L^1(0,\infty)$

$f_n(x) = e^{-nx} f(x)$

$f \in L^1$ dus $\int_0^\infty |f(x)|\,dx < \infty$

$e^{-nx} \leq 1$ voor $n \geq 0$ en $x \geq 0$, dus:

$|f_n(x)| = |e^{-nx} f(x)| \leq |f(x)|$ oftewel $f_n(x)$ wordt gedomineerd door een L^1-functie en we mogen dus limiet en integraal verwisselen:

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} e^{-nx} f(x) = 0$

Dus $\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx = \int_0^\infty 0\,dx = 0$

b $f_n(x) = \begin{cases} n^\alpha & \text{als } -\frac{1}{n} \le x \le \frac{1}{n} \\ 0 & \text{anders} \end{cases}$

$\int_{\mathbb{R}} f_n(x)\,dx = \int_{-\frac{1}{n}}^{\frac{1}{n}} n^\alpha\,dx = \frac{2}{n} n^\alpha = 2n^{\alpha-1}$

$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\,dx = \lim_{n\to\infty} 2n^{\alpha-1} = \begin{cases} 0 \text{ als } \alpha-1<0 \Leftrightarrow \alpha<1 \\ \cancel{2 \text{ als } \alpha-1=0 \Leftrightarrow \alpha=1} \\ +\infty \text{ als } \alpha-1>0 \Leftrightarrow \alpha>1 \end{cases}$

Dus $\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\,dx = +\infty$ voor $\alpha>1$.

Hier vindt, als $\alpha<1$, gedomineerde convergentie plaats, want $f_n(x)$ wordt gedomineerd door $g(x)=|x|^{-\alpha}$ en deze is L^1 op $[-1,1]$ voor $\alpha<1$.

3a $f: [-\pi,\pi] \to \mathbb{C}$

$f(t) = te^{it}$

$c_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-it}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} te^{it}e^{-it}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} t\,dt = 0$

$n\neq 1$:

$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} te^{it}e^{-int}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} te^{it(1-n)}\,dt =$

$\frac{1}{2\pi}\left(\left[t\frac{1}{i(1-n)}e^{it(1-n)}\right]_{t=-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{1}{i(1-n)}e^{it(1-n)}\,dt\right) =$

$\frac{1}{2\pi}\left(\frac{\pi}{i(1-n)}\cos(\pi(1-n)) - \frac{-\pi}{i(1-n)}\cos(-\pi(1-n)) - \left[\frac{1}{i^2(1-n)^2}e^{it(1-n)}\right]_{-\pi}^{\pi}\right) =$

$\frac{1}{2\pi}\left(\frac{2\pi}{i(1-n)}\cos(\pi(1-n)) + \frac{1}{(1-n)^2}\left[e^{it(1-n)}\right]_{-\pi}^{\pi}\right) =$

$\frac{1}{2\pi}\left(\frac{2\pi}{i(1-n)}(-1)^{1-n} + \frac{1}{(1-n)^2}\left(\cos(\pi(1-n)) - \cos(-\pi(1-n))\right)\right) =$

$\frac{1}{i(1-n)}(-1)^{1-n} = i\frac{-1}{1-n}(-1)^{1-n} = i\frac{(-1)^n}{1-n}$

b $x\sin x = \operatorname{Im}(xe^{ix}) = \operatorname{Im}\left(\sum_{n\neq1} i\frac{(-1)^n}{1-n}e^{inx}\right) =$

$\operatorname{Im}\left(\sum_{n\neq1} i\frac{(-1)^n}{1-n}(\cos nx + i\sin nx)\right) = \operatorname{Im}\left(\sum_{n\neq1} i\frac{(-1)^n}{1-n}\cos nx - \frac{(-1)^n}{1-n}\sin nx\right)$

$= \sum_{n\neq1}\frac{(-1)^n}{1-n}\cos nx$

3c $x \sin x = \sum_{n\neq 1} \frac{(-1)^n}{1-n} \cos nx = \sum_{n\leq -2} \frac{(-1)^n}{1-n} \cos nx + \sum_{n=-1} \frac{(-1)^n}{1-n} \cos nx +$

$\sum_{n=0} \frac{(-1)^n}{1-n} \cos nx + \sum_{n\geq 2} \frac{(-1)^n}{1-n} \cos nx = \sum_{n\leq 2} \frac{(-1)^n}{1-n} \cos nx - \frac{1}{2} \cos nx +$

$1 + \sum_{n\geq 2} \frac{(-1)^n}{1-n} \cos nx = 1 - \frac{1}{2} \cos nx + \sum_{n\geq 2} \left(\frac{(-1)^n}{1-n} \cos nx + \frac{(-1)^{-n}}{1+n} \cos(-nx) \right) =$

$1 - \frac{1}{2} \cos nx + \sum_{n=2}^{\infty} \frac{(-1)^n}{1-n} \cos nx + \frac{(-1)^n}{1+n} \cos nx =$

$1 - \frac{1}{2} \cos nx + \sum_{n=2}^{\infty} \left(\frac{(-1)^n}{1-n} + \frac{(-1)^n}{1+n} \right) \cos nx =$

$1 - \frac{1}{2} \cos nx + \sum_{n=2}^{\infty} \frac{(1+n)(-1)^n + (1-n)(-1)^n}{(1-n)(1+n)} \cos nx =$

$1 - \frac{1}{2} \cos nx + \sum_{n=2}^{\infty} \frac{2(-1)^n}{1-n^2} \cos nx = 1 - \frac{1}{2} \cos nx - 2\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1} \cos nx$

d $x \sin x = 1 - \frac{1}{2} \cos x - 2 \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1} \cos nx$

Vul in $x = \pi$:

$0 = 1 + \frac{1}{2} - 2 \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1} (-1)^n = \frac{3}{2} - 2 \sum_{n=2}^{\infty} \frac{(-1)^{2n}}{n^2-1} = \frac{3}{2} - 2 \sum_{n=2}^{\infty} \frac{1}{n^2-1}$

$2 \sum_{n=2}^{\infty} \frac{1}{n^2-1} = \frac{3}{2} \Rightarrow \sum_{n=2}^{\infty} \frac{1}{n^2-1} = \frac{3}{4}$

4a $f(t) = \begin{cases} 1 & \text{als } -1 \leq t \leq 1 \\ 0 & \text{anders} \end{cases}$

$g(t) = \begin{cases} 2-|t| & \text{als } -2 \leq t \leq 2 \\ 0 & \text{anders} \end{cases}$

$(f*f)(x) = \int_{\mathbb{R}} f(t) f(x-t)\,dt = \int_{-1}^{1} f(x-t)\,dt$

Voor $x<-2$ is $\int_{-1}^{1} f(x-t) = 0$ als $-1 \leq t \leq 1$

Voor $x>2$ is $f(x-t) = 0$ als $-1 \leq t \leq 1$.

Voor $-2 \leq x \leq 0$ geldt:

$(f*f)(x) = \int_{-1}^{1} f(x-t)\,dt = \int_{-1}^{1+x} 1\,dt = [t]_{t=-1}^{t=1+x} = 1+x+1 = 2+x$

$= 2 - |x|$ omdat $x \leq 0$

Voor $0 \leq x \leq 2$ geldt:

$(f*f)(x) = \int_{-1}^{1} f(x-t)\,dt = \int_{x-1}^{1} 1\,dt = [t]_{t=x-1}^{t=1} = 1-(x-1) = 2-x$

$= 2-|x|$ omdat $x \geq 0$

Dus geldt $(f*f)(x) = \begin{cases} 2-|x| & \text{als } -2 \leq x \leq 2 \\ 0 & \text{anders} \end{cases} = g(x)$

b $\hat{f}(x) = \int_{\mathbb{R}} e^{ixt} f(t)\,dt = \int_{-1}^{1} e^{ixt} = \left[\frac{1}{ix} e^{ixt}\right]_{t=-1}^{t=1} = \frac{1}{ix}(e^{ix} - e^{-ix})$

$= \frac{1}{ix}(2i \sin x) = \frac{-i}{x}(2i \sin x) = \frac{2 \sin x}{x}$ ✓

c $\hat{g}(x) = \widehat{f*f}\,(x) = (\hat{f} \cdot \hat{f})(x) = \frac{4 \sin^2 x}{x^2}$ ✓

d $\int_{\mathbb{R}} \frac{\sin^4 x}{x} dx = \frac{1}{16} \int_{\mathbb{R}} \left(\frac{4 \sin^2 x}{x^2}\right)^2 dx = \frac{1}{16} \int_{\mathbb{R}} |\hat{g}(x)|^2 dx = \frac{2\pi}{16} \int_{\mathbb{R}} |g(t)|^2 dt =$

$\frac{2\pi}{16} \int_{\mathbb{R}} g(t)^2 dt = \frac{2\pi}{16} \left(\int_{-2}^{0} (2+t)^2 dt + \int_{0}^{2} (2-t)^2 dt \right) =$

$\frac{2\pi}{16} \left(\left[2t + \frac{1}{2}t^2\right]_{t=-2}^{t=0} + \left[2t - \frac{1}{2}t^2\right]_{t=0}^{t=2} \right) =$

$\frac{2\pi}{16} \left(\cancel{4} - (-4 + \frac{1}{2} \cdot 4) + (4 - \frac{1}{2} \cdot 4) \right) = \frac{2\pi}{16} (4 - 2 + 4 - 2) = \frac{2\pi \cdot 4}{16} = \frac{\pi}{2}$

$\frac{2\pi}{3}$